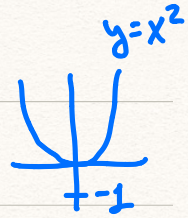
$$\mathbb{N} \supset \mathbb{Z} \supset \mathbb{Q} \supset \mathbb{R} \supset \mathbb{C}$$

We begin by introducing a new object i with the property

$$i^2 = -1 \qquad\qquad x^2 = -1$$



- Since $x^2 = -1$ has no solution in $\mathbb{R}$, $i \notin \mathbb{R}$.

Definition

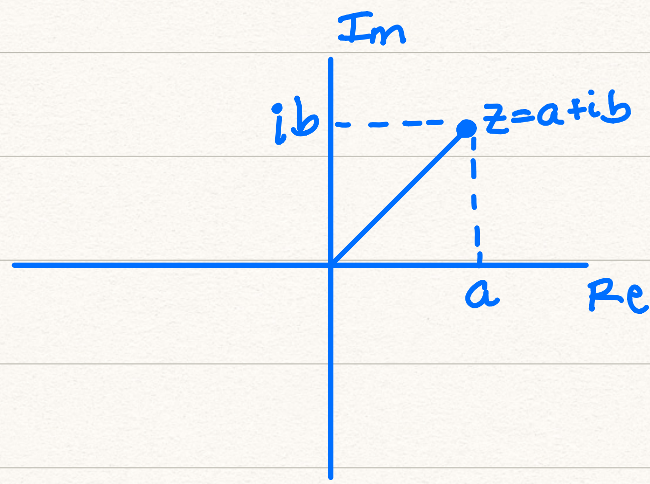
A complex number z is defined as

$$z = a + ib \qquad a, b \in \mathbb{R}.$$

The set of all complex numbers $\mathbb{C}$ is known as the complex plane.

- Each complex number $z = a + ib$ is identified with the point $(a,b) \in \mathbb{R}^2$.

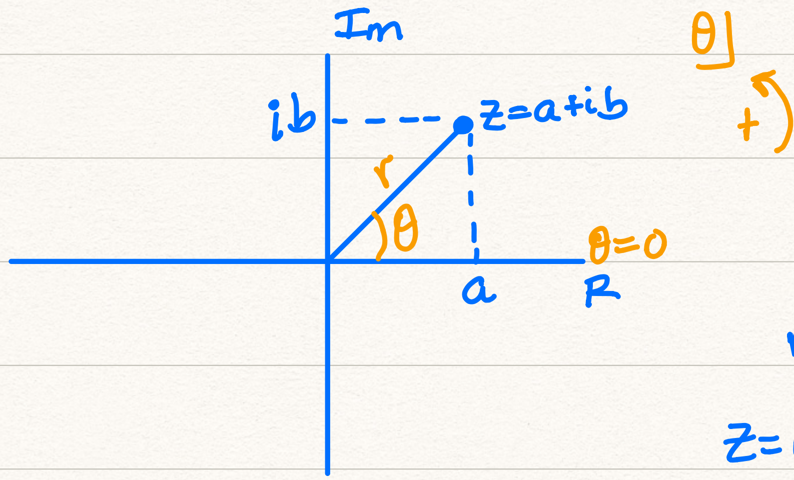
$\mathbb{C}$



- a is called the real part of z and b is called the imaginary part of z.

$$\mathrm{Re}\, z = a \qquad \mathrm{Im}\, z = b$$

Rectangular and polar representation



radians

$z=0$ $\quad$ θ not
$r=0$ $\quad$ defined

$$\boxed{\begin{array}{ll} r=\sqrt{a^2+b^2} & \theta=\arctan b/a \\ a=r\cos\theta & b=r\sin\theta \end{array}}$$

$0 \leq \theta < 2\pi$

- Rectangular : $z=a+ib$
- Polar : $z=r\cos\theta+ir\sin\theta$

$z=a+ib$
with
$b=0$

Definition $\quad z=a+ib=r\cos\theta+ir\sin\theta$ $\quad z\in\mathbb{R}$

Modulus : $|z|=r=\sqrt{a^2+b^2}$ $\quad |z|=\sqrt{a^2}$

argument : $\arg z=\theta=\arctan(b/a)$

Euler Formula

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \theta \in \mathbb{R}$$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$z = a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

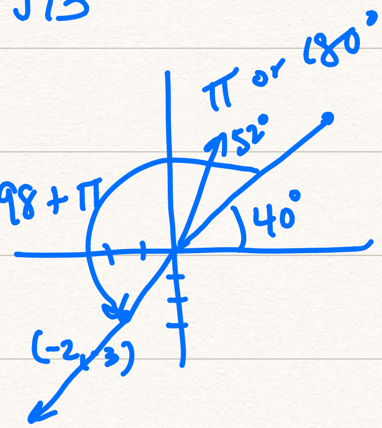
Example:

Express the following numbers in polar form:

(a) $-2-i3$ (b) $2+i3$ (c) $-2+i$ (d) $1-i3$

(a) $r = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

$$\theta = \arctan\left({}^{-3}/_{-2}\right) \approx \cancel{41}$$
$$\approx 221^\circ \approx 0.98 + \pi$$

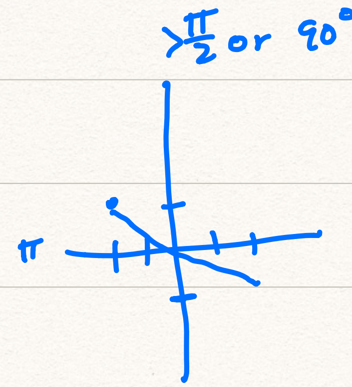


$$-2-i3 = \sqrt{13}\, e^{i(\arctan 3/2 + \pi)}$$

b) $2+i3=\sqrt{13}\,e^{i\arctan 3/2}$

(c) $r=\sqrt{(-2)^2+1}=\sqrt{5}$

$\theta=\arctan(-1/2)+\pi$

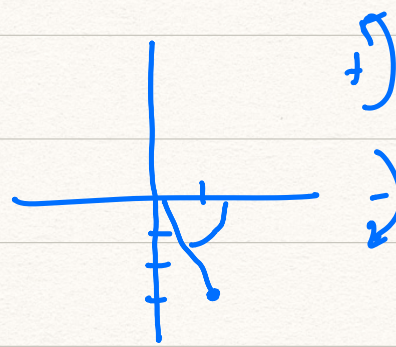


$-2+i=\sqrt{5}\,e^{i(\arctan(-1/2)+\pi)}$

(d) $1-i3$

$r=\sqrt{1^2+(-3)^2}=\sqrt{10}$

$\theta=\arctan(-3/1)$



$1-i3=\sqrt{10}\,e^{i\arctan(-3)}$

Example. Represent the following numbers in the complex plane and express them in rectangular form:

(a) $2e^{i\pi/3}$ (b) $4e^{-i3\pi/4}$ (c) $2e^{i\pi/2}$ (d) $3e^{-3\pi i}$

(e) $2e^{i4\pi}$ (f) $2e^{-i4\pi}$

(a) $2e^{i\pi/3} = 2(\cos \pi/3 + i \sin \pi/3)$

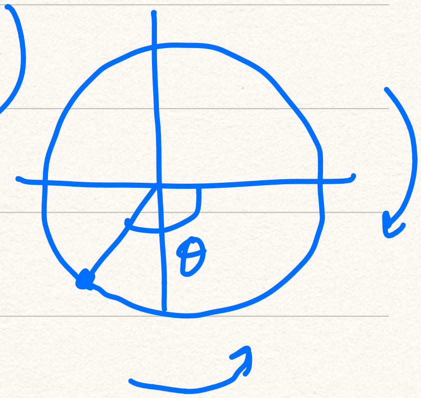
$= 2(1/2 + i\sqrt{3}/2)$

$= 1 + i\sqrt{3}$

(b) $4e^{-i3\pi/4} = 4(\cos(-3\pi/4) + i\sin(-\frac{3\pi}{4}))$

$= 4(-\sqrt{2}/2 - i\sqrt{2}/2)$

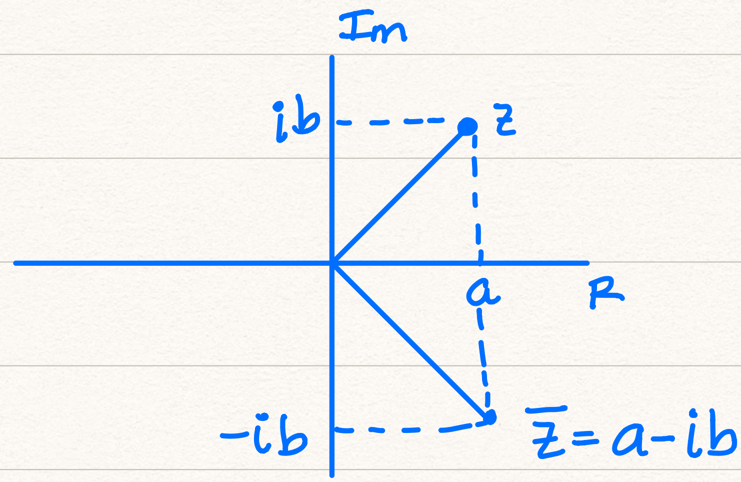
$= -2\sqrt{2} - i2\sqrt{2}$



The rest are exercises

$\frac{-3\pi}{4} + 2\pi = \frac{-3\pi + 8\pi}{4} = \frac{5\pi}{4}$

Conjugate of a complex number



Definition

The complex conjugate of a number $z=a+ib$ is the number $\bar{z}=a-ib$

- Polar : $\bar{z}=re^{-i\theta}$

- $z\bar{z}=(a+ib)(a-ib)=a^2+b^2=r^2=|z|^2$

$$a^2-\cancel{iab}+\cancel{iab}+(ib)^2$$

$i^2=-1$ $\quad = a^2+b^2$

$$|z|=\sqrt{z\bar{z}}$$

- $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

$e^{i\theta}+e^{-i\theta} = \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta} = 2\cos\theta$

- $z+\bar{z} = 2\operatorname{Re} z$, $z-\bar{z} = 2i\operatorname{Im} z$

$a+\cancel{ib}+a-\cancel{ib} = 2a$ $\quad z = e^{i\theta}$

<u>Example</u>. For $z_1 = 2e^{i\pi/4}$ and $z_2 = 8e^{i\pi/3}$,

find (a) $2z_1 - z_2$ (b) $\dfrac{1}{z_1}$ (c) $\dfrac{z_1}{z_2^2}$ (d) $\sqrt[3]{z_2}$ $z_2^{1/3}$

(a) $2z_1 - z_2 = 4e^{i\pi/4} - 8e^{i\pi/3}$

$= 4\left(\sqrt{2}/2 + i\sqrt{2}/2\right) - 8\left(1/2 + i\sqrt{3}/2\right)$

$= 2\sqrt{2} + i2\sqrt{2} - 4 - i4\sqrt{3}$

$= (2\sqrt{2}-4) + i(2\sqrt{2} - 4\sqrt{3})$

(b) $\frac{1}{z_1} = \frac{1}{2e^{i\pi/4}} = \frac{1}{2} e^{-i\pi/4}$

$re^{i\theta}$

$\uparrow$

$r \quad \theta = -\pi/4$

$\frac{3\pi - 8\pi}{12}$

c) $\frac{z_1}{z_2^2} = \frac{2e^{i\pi/4}}{(8e^{i\pi/3})^2} = \frac{2e^{i\pi/4}}{64e^{i2\pi/3}} = \frac{1}{32} e^{i(\pi/4 - 2\pi/3)}$

$= \frac{1}{32} e^{-i5\pi/12}$

__Example__. For $z_1 = 3 + i4$ and $z_2 = 2 + i3$ determine $z_1 z_2$ and z_1 / z_2

$$\frac{z_1}{z_2} = \frac{3+i4}{2+i3} \cdot \frac{2-i3}{2-i3} = \frac{(3+i4)(2-i3)}{4+9}$$

$a + ib$

$re^{i\theta}$

$n^2 - n$ is even

$n(n-1)$

if n is odd, then $n-1$ is even so n^2-n is even.

if n is even, then the result is always even.

if n is a perfect square, then $\sqrt{n} \in \mathbb{Q}$

remember proof 1.9(b)

(b) $\sqrt{n} \notin \mathbb{Q}$ if n is not a perfect square (HINT: write $n = k^2 r$, where r does not contain any square factor),

If n is not a perfect square, then at least one of its factors is not a square. So we can write $n = k^2 r$ where r does not contain any square factors.

Now, we argue by contradiction. Suppose that n is not a perfect square and $\sqrt{n} \in \mathbb{Q}$.
Then we can write $\sqrt{n} = p/q$, $p, q \in \mathbb{N}$, where p and q have no common factors (p/q is in its simplest form).
Then $n = p^2/q^2 = k^2 r$ or, equivalently, $r = \dfrac{p^2}{q^2 k^2}$. But this is imposible because r does not contain square factors. Hence, $\sqrt{n} \notin \mathbb{Q}$

Roots of complex numbers

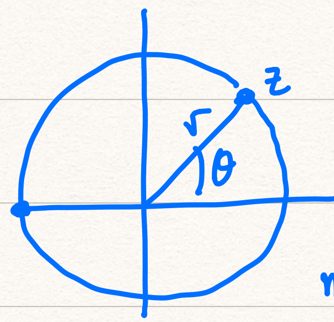
$m \in \mathbb{Z}$

$z = a + ib = re^{i\theta + 2\pi i m}$ $\quad r = \sqrt{a^2+b^2}$ $\quad$ arctan

$\tan\theta = \frac{b}{a}$ $\quad \theta = \tan^{-1}\frac{b}{a}$

$z^\alpha = (re^{i\theta})^\alpha = r^\alpha e^{i\theta\alpha + 2\pi i \alpha m}$

$\alpha \in \mathbb{R}$ $\quad \sqrt{z} = z^{1/2}$ $\quad \frac{5\pi}{3} \approx 300^\circ$



$x^3 + 1 = 0$

$m=1: \frac{\pi}{3}i + \frac{2\pi}{3}i$

$m=2: \frac{\pi}{3}i + \frac{4\pi}{3}i$

$r = 1$

$(-1)^{1/3} = \left(e^{\pi i + 2\pi i m}\right)^{1/3} = e^{\pi/3\, i + \frac{2\pi}{3} i m}$

$(-1)^{1/3} = -1$